Schwartz 3.1 (a)

$x_\mu \to \Lambda_{\mu\nu} x_\nu$, Let $\Lambda_{\mu\nu} = 1 + \varepsilon_{\mu\nu}$, then

$x_\mu \to x_\mu + \varepsilon_{\mu\nu} x_\nu$

$$\mathcal{L}(x_\mu + \varepsilon_{\mu\nu} x_\nu) \simeq \mathcal{L}(x_\mu) + \varepsilon_{\mu\nu} x_\nu \left(\partial_\mu \mathcal{L}\right)\Big|_{x_\mu} + \cdots$$

$$\delta \mathcal{L} = \mathcal{L}(x_\mu + \varepsilon_{\mu\nu} x_\nu) - \mathcal{L}(x_\mu) \simeq \varepsilon_{\mu\nu} x_\nu \partial_\mu \mathcal{L}.$$

$$\Rightarrow \quad \frac{\delta \mathcal{L}}{\delta \varepsilon_{\mu\nu}} = x_\nu \partial_\mu \mathcal{L}.$$

$$\frac{\delta \mathcal{L}}{\delta \varepsilon_{\mu\nu}} = \sum_n \frac{\partial \mathcal{L}}{\partial \phi_n} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial(\partial_\alpha \phi_n)}{\partial \varepsilon_{\mu\nu}}$$

$$= \sum_n \frac{\partial \mathcal{L}}{\partial \phi_n} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} + \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}}\right] - \left[\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)}\right] \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}}$$

$$= \sum_n \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)}\right]}_{\text{vanish on eqm.}} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} + \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}}\right]$$

$$= \sum_n \partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}}\right]$$

$$\Rightarrow \quad \partial_\alpha \left\{ \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} \right\} = \frac{\delta \mathcal{L}}{\delta \varepsilon_{\mu\nu}} = x_\nu \partial_\mu \mathcal{L},$$

Let the summation on ϕ_n be implicit,

$$\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} \right] - x_\nu \partial_\mu \mathcal{L} = 0$$

$$\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} - g_{\mu\alpha} x_\nu \mathcal{L} \right] = 0.$$

$$\Rightarrow \quad \partial_\alpha K_{\mu\nu\alpha} = 0, \qquad K_{\mu\nu\alpha} = \text{above}.$$

Compute $\dfrac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}}$, $\quad \phi_n(x_\mu + \varepsilon_{\mu\nu} x_\nu) = \phi_n|_{x_\mu} + \varepsilon_{\mu\nu} x_\nu \partial_\mu \phi_n|_{x_\mu} + \ldots$

$$\Rightarrow \quad \frac{\partial \phi_n}{\partial \varepsilon_{\mu\nu}} = x_\nu \partial_\mu \phi_n.$$

$$\Rightarrow \quad K_{\mu\nu\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} x_\nu \partial_\mu \phi_n - g_{\mu\alpha} x_\nu \mathcal{L}$$

$$= x_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_n)} \partial_\mu \phi_n - g_{\mu\alpha} \mathcal{L} \right]$$

It satisfies $\partial_\alpha K_{\mu\nu\alpha} = 0$.

To get this in the form of $\partial_\mu K_{\mu\nu\alpha} = 0$, we switch some indices:

$$K_{\mu\nu\alpha} = x_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_n)} \partial_\mu \phi_n - g_{\mu\nu} \mathcal{L} \right]$$

$$= \boxed{x_\alpha T_{\mu\nu}}$$ with some relabeling on indices.

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